ON LOCAL AND BOUNDARY BEHAVIOR OF MAPPINGS IN METRIC SPACES

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Abstract

Open discrete mappings with a modulus condition in metric spaces are considered. Some results related to local behavior of mappings as well as theorems about continuous extension to a boundary are proved.

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1 Introduction

The paper is devoted to the study of quasiregular mappings and their natural generalizations investigated long time, see e.g. [AC], [Cr_1]–[Cr_2], [Gol_1]–[Gol_2], [GRSY], [IM], [MRSY], [MRV_1]–[MRV_3], [Re], [Ri], [Vu] and further references therein. We also refer to work of Novosibirsk mathematical school, see [Vo_1]–[UV].

As known, boundary and local behavior of quasiregular mappings in $\mathbb{R}^n$ are the main subjects of investigation in many works, see [Ge], [Na], [MRV_2], [Ri], [Va_1], [Va_2] etc. It should also be noted a large number of works by mapping with finite distortion in this context, see e.g. [Cr_1]–[Cr_2], [Gol_1]–[Gol_2], [GRSY], [IM], [HK], [MRSY] and [Ra]. Besides that, we refer to works, where mappings obeying modular inequalities are studied, see [IR_1]–[IR_2], [RS], [Sm] and [Sev_1]–[Sev_3]. Mappings mentioned above are called Q-mappings, and were introduced by O. Martio together with V. Ryazanov, U. Srebro and E. Yakubov, see [MRSY].

Now we return to [Sev_1]–[Sev_3]. Local behavior of mappings satisfying modular inequalities is studied in [Sev_1]. In particular, we have proved here that a family of mappings mentioned above is equicontinuous provided that characteristic of quasiconformality $Q(x)$ has a finite mean oscillation at the corresponding point. In [Sev_2], we have proved that sets of zero modulus with weight Q (in particular, isolated singularities) are removable for discrete open Q-mappings if the function $Q(x)$ has finite mean oscillation or a logarithmic

singularity of order not exceeding $n-1$ on the corresponding set. The problem of extension of mappings $f: D\to\mathbb{R}^n$ with modular condition to the boundary of a domain D has been investigated in [Sev$_3$]. Under certain conditions imposed on a measurable function $Q(x)$ and the boundaries of the domains D and $D'=f(D)$ we show that an open discrete mapping $f: D\to\mathbb{R}^n$ with quasiconformality characteristic $Q(x)$ can be extended to the boundary ∂D by continuity.

Now we continue studying mappings satisfying modular conditions. In the present paper we show that some results from [Sev$_1$]–[Sev$_3$] holds not only in $\mathbb{R}^n$, but in metric spaces, also. Here we assume that mapping f is not injective, as rule, however, f is open and discrete. In addition, we need require the existence of maximal liftings of curves under mapping f. Note that the openness and discreteness of f in $\mathbb{R}^n$ implies the existence of maximal liftings of curves (see [Ri, Ch. 3.II]).

2 On equicontinuity of homeomorphisms between metric spaces

Let us give some definitions. Recall, for a given continuous path $\gamma:[a,b]\to X$ in a metric space (X,d), that its length is the supremum of the sums

$$\sum_{i=1}^{k} d(\gamma(t_i),\gamma(t_{i-1}))$$

over all partitions $a=t_0\leqslant t_1\leqslant\ldots\leqslant t_k=b$ of the interval $[a,b]$. The path γ is called *rectifiable* if its length is finite.

Given a family of paths Γ in X, a Borel function $\varrho: X\to[0,\infty]$ is called *admissible* for Γ, abbr. $\varrho\in\operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma}\varrho\,ds\ \geqslant\ 1 \tag{2.1}$$

for all (locally rectifiable) $\gamma\in\Gamma$. Everywhere further, for any sets E,F, and G in X, we denote by $\Gamma(E,F,G)$ the family of all continuous curves $\gamma:[0,1]\to X$ such that $\gamma(0)\in E$, $\gamma(1)\in F$, and $\gamma(t)\in G$ for all $t\in(0,1)$. For $x_0\in X$ and $r>0$, the ball $\{x\in X: d(x,x_0)<r\}$ is denoted by $B(x_0,r)$. Everywhere further (X,d,μ) and (X',d',μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly.

An open set any two points of which can be connected by a curve is called a domain in X. The modulus of a family of curves Γ in a domain G of finite Hausdorff dimension $\alpha\geqslant 2$ from X is defined by the equality

$$M_{\alpha}(\Gamma)=\inf_{\rho\in\operatorname{adm}\Gamma}\int\limits_{G}\varrho^{\alpha}(x)d\mu(x)\,. \tag{2.2}$$

In the case of the path family $\Gamma' = f(\Gamma)$ we take the Hausdorff dimension α' of the domain G'.

A family of paths Γ_1 in X is said to be *minorized* by a family of paths Γ_2 in X, abbr. $\Gamma_1 > \Gamma_2$, if, for every path $\gamma_1 \in \Gamma_1$, there is a path $\gamma_2 \in \Gamma_2$ such that γ_2 is a restriction of γ_1. In this case

$$\Gamma_1 > \Gamma_2 \quad \Rightarrow \quad M_\alpha(\Gamma_1) \le M_\alpha(\Gamma_2) \tag{2.3}$$

(see [Fu, Theorem 1]).

Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 2$ in spaces (X, d, μ) and (X', d', μ'), and let $Q : G \to [0, \infty]$ be a measurable function. Set $S(x_0, r_i) = \{x \in X : d(x_0, x) = r_i\}$, $i = 1, 2$, $0 < r_1 < r_2 < \infty$. Following to [MRSY, Ch. 7], we say that a mapping $f : G \to G'$ is a ring Q-mapping at a point $x_0 \in G$ if the inequality

$$M_{\alpha'}(f(\Gamma(S_1, S_2, A)) \leqslant \int\limits_{A\cap G} Q(x)\eta^\alpha(d(x, x_0))d\mu(x) \tag{2.4}$$

holds for any ring

$$A = A(x_0, r_1, r_2) = \{x \in X : r_1 < d(x, x_0) < r_2\}, \quad 0 < r_1 < r_2 < \infty\,, \tag{2.5}$$

and any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)dr \geqslant 1\,. \tag{2.6}$$

A family $\mathcal{F}$ of continuous functions f defined on some metric space (X, d) with values in another metric space (Y, d') is called equicontinuous at a point $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d'(f(x_0), f(x)) < \varepsilon$ for all $f \in \mathcal{F}$ and all x such that $d(x_0, x) < \delta$. The family is equicontinuous if it is equicontinuous at each point of X. Thus, by the well-known Ascoli theorem, normality is equivalent to equicontinuity on compact sets of the mappings in $\mathcal{F}$.

Let G be a domain in a space (X, d, μ). Similarly to [IR$_1$], we say that a function $\varphi : G \to \mathbb{R}$ has *finite mean oscillation at a point* $x_0 \in \overline{G}$, abbr. $\varphi \in FMO(x_0)$, if

$$\overline{\lim_{\varepsilon\to 0}}\ \frac{1}{\mu(B(x_0, \varepsilon))} \int\limits_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon|\ d\mu(x) < \infty \tag{2.7}$$

where

$$\overline{\varphi}_\varepsilon = \frac{1}{\mu(B(x_0, \varepsilon))} \int\limits_{B(x_0,\varepsilon)} \varphi(x)\ d\mu(x)$$

is the mean value of the function $\varphi(x)$ over the set

$$B(x_0, \varepsilon) = \{x \in G : d(x, x_0) < \varepsilon\}$$

with respect to the measure μ. Here the condition (2.7) includes the assumption that φ is integrable with respect to the measure μ over the set $B(x_0, \varepsilon)$ for some $\varepsilon > 0$.

Following [He, section 7.22], given a real-valued function u in a metric space X, a Borel function $\rho\colon X \to [0, \infty]$ is said to be an *upper gradient* of a function $u : X \to \mathbb{R}$ if $|u(x) - u(y)| \leqslant \int\limits_{\gamma} \rho\,|dx|$ for each rectifiable curve γ joining x and y in X. Let (X, μ) be a metric measure space and let $1 \leqslant p < \infty$. We say that X admits *a* $(1; p)$*-Poincare inequality* if there is a constant $C \geqslant 1$ such that

$$\frac{1}{\mu(B)} \int\limits_{B} |u - u_B| d\mu(x) \leqslant C \cdot (\operatorname{diam} B) \left(\frac{1}{\mu(B)} \int\limits_{B} \rho^p d\mu(x) \right)^{1/p}$$

for all balls B in X, for all bounded continuous functions u on B, and for all upper gradients ρ of u. Metric measure spaces where the inequalities

$$\frac{1}{C} R^n \leqslant \mu(B(x_0, R)) \leqslant C R^n$$

hold for a constant $C \geqslant 1$, every $x_0 \in X$ and all $R < \operatorname{diam} X$, are called *Ahlfors n-regular.* As known, Ahlfors n-regular spaces have Hausdorff dimension α (see e.g. [He, p. 61–62]). A domain G in a topological space T is called *locally connected at a point* $x_0 \in \partial G$ if, for every neighborhood U of the point x_0, there is its neighborhood $V \subset U$ such that $V \cap G$ is connected (see [Ku, I.6, § 49]).

Theorem 2.1. *Let G be a domain in a locally connected and a locally compact metric space (X, d, μ) with a finite Hausdorff dimension $\alpha \geqslant 2$, and let (X', d', μ') be an Ahlfors α'-regular metric space which supports $(1; \alpha')$-Poincare inequality. Let $B_R \subset X'$ is a fixed ball of a radius R. Denote $\mathfrak{R}_{x_0, Q, B_R, \delta}(G)$ a family of ring Q-homeomorphisms $f\colon G \to B_R \setminus K_f$ at $x_0 \in G$ with $\sup\limits_{x,y \in K_f} d'(x, y) \geqslant \delta > 0$, where $K_f \subset B_R$ is some continuum. Then $\mathfrak{R}_{x_0, Q, B_R, \delta}(G)$ is equicontinuous at $x_0 \in G$ whenever $Q \in FMO(x_0)$.*

The following lemma can be useful under investigations related to equicontinuity of families of mappings.

Lemma 2.1. *Let G be a domain in a metric space (X, d, μ) with a finite Hausdorff dimension $\alpha \geqslant 2$, and let (X', d', μ') be a metric space with a finite Hausdorff dimension $\alpha' \geqslant 2$. Let $f\colon G \to X'$ be a ring Q-mapping at $x_0 \in G$, and let $\varepsilon_0 > 0$ be such that $\overline{B(x_0, \varepsilon_0)} \subset G$. Assume that*

$$\int\limits_{\varepsilon < d(x, x_0) < \varepsilon_0} Q(x) \cdot \psi_{\varepsilon}^{\alpha}(d(x, x_0))\, d\mu(x) \leqslant F(\varepsilon, \varepsilon_0) \qquad \forall\, \varepsilon \in (0, \varepsilon_0'), \tag{2.8}$$

for some $\varepsilon_0' \in (0, \varepsilon_0)$ and some family of nonnegative Lebesgue measurable functions $\{\psi_{\varepsilon}(t)\}$, $\psi_{\varepsilon}\colon (\varepsilon, \varepsilon_0) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0')$, where где $F(\varepsilon, \varepsilon_0)$ is some function, and

$$0 < I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{\varepsilon}(t) dt < \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0'). \tag{2.9}$$

Denote $S_1 = S(x_0,\varepsilon)$, $S_2 = S(x_0,\varepsilon_0)$ *and* $A = \{x \in G : \varepsilon < d(x,x_0) < \varepsilon_0\}$. *Then*

$$M_{\alpha'}(f(\Gamma(S_1,S_2,A)) \leqslant F(\varepsilon,\varepsilon_0)/I^{\alpha}(\varepsilon,\varepsilon_0) \qquad \forall\, \varepsilon \in (0,\, \varepsilon_0^{\prime})\ . \tag{2.10}$$

Proof. Set $\eta_\varepsilon(t) = \psi_\varepsilon(t)/I(\varepsilon,\varepsilon_0)$, $t \in (\varepsilon,\, \varepsilon_0)$. We observe that $\int\limits_{\varepsilon}^{\varepsilon_0} \eta_\varepsilon(t)\, dt = 1$ for $\varepsilon \in (0, \varepsilon_0^{\prime})$. Now, from the definition of ring Q-mapping at x_0, and from (2.8), we obtain (2.10). □

The following statement holds (see [AS, Proposition 4.7]).

Proposition 2.1. *Let* X *be a* α*-Ahlfors regular metric measure space that supports* $(1;\alpha)$*-Poincare inequality for some* $\alpha > 1$. *Let* E *and* F *be continua contained in a ball* $B(x_0,R)$. *Then*

$$M_\alpha(\Gamma(E,F,X)) \geqslant \frac{1}{C}\cdot\frac{\min\{\operatorname{diam} E, \operatorname{diam} F\}}{R}$$

for some constant $C > 0$.

The following lemma provides the main tool for establishing equicontinuity in the most general situation.

Lemma 2.2. *Let* G *be a domain in a locally connected and locally compact metric space* (X,d,μ) *with a finite Hausdorff dimension* $\alpha \geqslant 2$, *and let* (X',d',μ') *be an Ahlfors* α'*-regular metric space which supports* $(1;\alpha')$*-Poincare inequality. Let* $r_0 > 0$ *be such that* $\overline{B(x_0,\varepsilon_0)} \subset G$ *and* $0 < \varepsilon_0 < r_0$. *Assume that,* (2.8) *holds for some* $\varepsilon_0^{\prime} \in (0,\varepsilon_0)$, *and for some family of nonnegative Lebesgue measurable function* $\{\psi_\varepsilon(t)\}$, $\psi_\varepsilon\colon (\varepsilon,\varepsilon_0) \to [0,\infty]$, $\varepsilon \in (0,\varepsilon_0^{\prime})$, *where* $F(\varepsilon,\varepsilon_0)$ *is some function for which* $F(\varepsilon,\varepsilon_0) = o(I^{\alpha}(\varepsilon,\varepsilon_0))$, *and* $I(\varepsilon,\varepsilon_0)$ *is defined in* (2.9).

Let $B_R \subset X'$ *be a fixed ball of a radius* R. *Denote* $\mathfrak{R}_{x_0,Q,B_R,\delta}(G)$ *a family of all ring* Q*-homeomorphisms* $f\colon G \to B_R \setminus K_f$ *at* $x_0 \in G$ *with* $\sup\limits_{x,y\in K_f} d'(x,y) \geqslant \delta > 0$, *where* $K_f \subset B_R$ *is a fixed continuum. Now,* $\mathfrak{R}_{Q,x_0,B_R,\delta}(G)$ *is equicontinuous at* x_0.

Proof. Fix $x_0 \in G$, $f \in \mathfrak{R}_{x_0,Q,B_R,\delta}(G)$. Since X is locally connected and locally compact space, we can find a sequence $B(x_0,\varepsilon_k)$, $k = 0,1,2,\ldots$, $\varepsilon_k \to 0$ as $k \to \infty$, such that $V_{k+1} \subset \overline{B(x_0,\varepsilon_k)} \subset V_k$, where V_k are continua in G. Observe that $f(V_k)$ are K_f continua in B_R, in fact, $f(V_k)$ is a continuum as continuous image of a continuum (see e.g. [Ku, Theorem 1.III.41 and Theorem 3.I.46]). Now, by Proposition 2.1 we obtain that

$$M_{\alpha'}(K_f, f(V_k), X')) \geqslant \frac{1}{C}\cdot\frac{\min\{\operatorname{diam} K_f, \operatorname{diam} f(V_k)\}}{R} \tag{2.11}$$

at some $C > 0$. Note that $\gamma \in \Gamma(K_f, f(V_k), X')$ does not fully belong to $f(B(x_0,\varepsilon_0))$ as well as $X' \setminus f(B(x_0,\varepsilon_0))$, so there exists $y_1 \in |\gamma| \cap f(S(x_0,\varepsilon_0))$ (see [Ku, Theorem 1, § 46, section. I]). Let $\gamma : [0,1] \to X'$ and $t_1 \in (0,1)$ be such that $\gamma(t_1) = y_1$. Without loss of generalization, we can consider that $|\gamma|_{[0,t_1)}| \in f(B(x_0,\varepsilon_0))$. Denote $\gamma_1 := \gamma|_{[0,t_1)}$, and set $\alpha_1 = f^{-1}(\gamma_1)$. Observe that $|\alpha_1| \in B(x_0,\varepsilon_0)$. Moreover, note that α_1 does not wholly belong to $\overline{B(x_0,\varepsilon_{k-1}}$ as well as to $X \setminus \overline{B(x_0,\varepsilon_{k-1})}$. Thus, there exists $t_2 \in (0,t_1)$ with

$\alpha_1(t_2)\in S(x_0,\varepsilon_{k-1})$ (see [Ku, Theorem 1, § 46, Section. I]). Without loss of generality, we can consider that $|\alpha_{[t_2,t_1]}|\in X\setminus\overline{B(x_0,\varepsilon_{k-1})}$. Set $\alpha_2=\alpha_1|_{[t_2,t_1]}$. Observe that $\gamma_2:=f(\alpha_2)$ is a subcurve of γ. From saying above,

$$\Gamma(K_f,f(V_k),X')>\Gamma(f(S(x_0,\varepsilon_{k-1})),f(S(x_0,\varepsilon_0)),f(A)))\,,$$

where $A=\{x\in X:\varepsilon_{k-1}<d(x,x_0)<\varepsilon_0\}$, whence by (2.3)

$$M_{\alpha'}(\Gamma(K_f,f(V_k),X'))\leqslant M_{\alpha'}(\Gamma(f(S(x_0,\varepsilon_{k-1})),f(S(x_0,\varepsilon_0)),f(A)))\,. \tag{2.12}$$

By (2.11) and (2.12), we conclude that

$$M_{\alpha'}(\Gamma(f(S(x_0,\varepsilon_{k-1})),f(S(x_0,\varepsilon_0)),f(A)))\geqslant\frac{1}{C}\cdot\frac{\min\{\operatorname{diam}K_f,\operatorname{diam}f(V_k)\}}{R}\,. \tag{2.13}$$

From other hand, by Lemma 2.1 and by $F(\varepsilon,\varepsilon_0)=o(I^{\alpha}(\varepsilon,\varepsilon_0))$, it follows that

$$M_{\alpha'}(\Gamma(f(S(x_0,\varepsilon_{k-1})),f(S(x_0,\varepsilon_0)),f(A)))\to 0$$

as $k\to\infty$. Therefore, for every $\sigma>0$ there exists $k_0\in\mathbb{N}=k_0(\sigma)$ such that

$$M_{\alpha'}(\Gamma(f(S(x_0,\varepsilon_{k-1})),f(S(x_0,\varepsilon_0)),f(A)))<\sigma$$

for every $k\geqslant k_0$. Now, by (2.13), it follows that

$$\min\{\operatorname{diam}K_f,\operatorname{diam}f(V_k)\}<\sigma \tag{2.14}$$

for $k\geqslant k_0$. Since $\operatorname{diam}K_f\geqslant\delta>0$ for every f, we obtain that

$$\min\{\operatorname{diam}K_f,\operatorname{diam}f(V_k)\}=\operatorname{diam}f(V_k)$$

for every $k\geqslant k_1(\sigma)$. Now, by (2.14)

$$\operatorname{diam}f(V_k)<\sigma \tag{2.15}$$

for every $k\geqslant k_1(\sigma)$. Since $V_{k+1}\subset\overline{B(x_0,\varepsilon_k)}\subset V_k$, the inequality (2.15) holds in $\overline{B(x_0,\varepsilon_k)}$ as $k\geqslant k_1(\sigma)$. Set $\varepsilon(\sigma):=\varepsilon_{k_1}$. Finally, given $\sigma>0$ there exists $\varepsilon(\sigma)>0$ such that $d'(f(x),f(x_0))<\sigma$ as $d(x,x_0)<\varepsilon(\sigma)$. So, $\mathfrak{R}_{Q,x_0,B_R,\delta}(G)$ is equicontinuous at x_0. $\Box$

The following statement can be found in [RS, Lemma 4.1].

Proposition 2.2. *Let G be a domain Ahlfors α-regular metric space (X,d,μ) at $\alpha\geqslant 2$. Assume that $x_0\in\overline{G}$ and $Q:G\to[0,\infty]$ belongs to $FMO(x_0)$. If*

$$\mu(G\cap B(x_0,2r))\leqslant\gamma\cdot\log^{\alpha-2}\frac{1}{r}\cdot\mu(G\cap B(x_0,r)) \tag{2.16}$$

for some $r_0>0$ and every $r\in(0,r_0)$, then Q satisfies (2.8), where $G(\varepsilon):=F(\varepsilon,\varepsilon_0)/I^n(\varepsilon,\varepsilon_0)$ obeying : $G(\varepsilon)\to 0$ as $\varepsilon\to 0$, and $\psi_\varepsilon(t)\equiv\psi(t):=\frac{1}{t\log\frac{1}{t}}$.

Proof of the Theorem 2.1 *follows from Lemma* 2.2 *and Proposition* 2.2. $\Box$

Taking into account [RS, Corollary 4.1], by Lemma 2.2, we obtain the following.

Corollary 2.1. *A conclusion of Theorem* 2.1 *holds, if instead of condition $Q\in FMO(x_0)$ we require that*

$$\limsup_{\varepsilon\to 0}\frac{1}{\mu(B(x_0,\varepsilon))}\int\limits_{B(x_0,\varepsilon)}Q(x)\,d\mu(x)<\infty\,.$$

3 Equicontinuity of open discrete mappings

In this section we prove a result similar to Theorem 2.1, where instead of homeomorphisms are considered open discrete mappings. However, in this case we have to require the following additional condition: the mapping should have a maximal lifting relative to an arbitrary curve. To give a definition.

Let $D\subset X$, $f:D\rightarrow X^{\,\prime}$ be a discrete open mapping, $\beta:[a,\,b)\rightarrow X^{\,\prime}$ be a curve, and $x\in\,f^{-1}\left(\beta(a)\right)$. A curve $\alpha:[a,\,c)\rightarrow D$ is called a *maximal f-lifting* of β starting at x, if (1) $\alpha(a)=x\,;$ (2) $f\circ\alpha=\beta|_{[a,\,c)};$ (3) for $c<c'\le b$, there is no curves $\alpha':[a,\,c')\rightarrow D$ such that $\alpha=\alpha'|_{[a,\,c)}$ and $f\circ\alpha'=\beta|_{[a,\,c')}$. In the case $X=X^{\,\prime}={\Bbb R}^n$, the assumption on f yields that every curve β with $x\in f^{\,-1}\left(\beta(a)\right)$ has a maximal f-lifting starting at x (see [Ri, Corollary II.3.3], [MRV$_3$, Lemma 3.12]).

Consider the condition

A : for all $\beta:[a,\,b)\rightarrow X^{\,\prime}$ and $x\in f^{\,-1}\left(\beta(a)\right)$, a mapping f has a maximal f-lifting starting at x.

Given $x_0\in D$ and $0<\varepsilon<\varepsilon_0$, let $A=A(x_0,\varepsilon,\varepsilon_0)$ be defined in (2.5), let $S_i=S(x_0,r_i)$ be sphere centered at x_0 of a radius r, and let $Q\colon D\rightarrow[0,\infty]$ be a measurable function. The following lemma holds.

Lemma 3.1. *Let G be a domain in a metric space (X,d,μ) with a finite Hausdorff dimension $\alpha\geqslant 2$, and let $(X^{\,\prime},d^{\,\prime},\mu^{\,\prime})$ be a metric space which has a finite Hausdorff dimension $\alpha^{\prime}\geqslant 2$. Let $f\colon G\rightarrow X^{\,\prime}$ be a ring Q-mapping at $x_0\in G$, and let $0<\varepsilon_0<\mathrm{dist}\,(x_0,\partial D)$ be such that $\overline{B(x_0,\varepsilon_0)}$ is compactum in D.*

Assume that

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0}Q(x)\cdot\psi_{\varepsilon}^{\alpha}(d(x,x_0))\,d\mu(x)\leqslant F(\varepsilon,\varepsilon_0)\qquad\forall\,\varepsilon\in(0,\varepsilon_0^{\prime})\tag{3.1}$$

holds for some $\varepsilon_0^{\prime}\in(0,\varepsilon_0)$, and some family of nonnegative Lebesgue measurable functions $\{\psi_{\varepsilon}(t)\}$, $\psi_{\varepsilon}\colon(\varepsilon,\varepsilon_0)\rightarrow[0,\infty]$, $\varepsilon\in(0,\varepsilon_0^{\prime})$, where $F(\varepsilon,\varepsilon_0)$ is some function, and (2.9) holds. If f satisfies the condition $\mathbf{A}$, then

$$M_{\alpha^{\prime}}(\Gamma(f(\overline{B(x_0,\varepsilon)}),\partial f(B(x_0,\varepsilon_0)),X^{\,\prime}))\leqslant F(\varepsilon,\varepsilon_0)/I^{\,\alpha}(\varepsilon,\varepsilon_0)\qquad\forall\,\varepsilon\in(0,\,\varepsilon_0^{\prime})\,.\tag{3.2}$$

Proof. We can assume that $\Gamma:=\Gamma(f(\overline{B(x_0,\varepsilon)}),\partial f(B(x_0,\varepsilon_0)),X^{\,\prime})\neq\varnothing$.

Now $\partial f(B(x_0,\varepsilon_0))\neq\varnothing$. Let Γ^* be a family of maximal f-liftings of Γ started at $\overline{B(x_0,\varepsilon)}$. Given a curve $\beta:[0,1)\rightarrow X^{\,\prime}$, $\beta\in\Gamma$, we show that it's maximal lifting $\alpha:[0,c)\rightarrow X$ satisfies the condition: $d(\alpha(t),S(x_0,\varepsilon_0))\rightarrow 0$ as $t\rightarrow c-0$.

Assume the contrary, i.e., there exists $\beta\colon [a,\,b) \to X'$ from Γ for which it's maximal lifting $\alpha\colon [a,\,c) \to B(x_0,\varepsilon_0)$ satisfies the condition $d(|\alpha|, \partial B(x_0,\varepsilon_0)) = \delta_0 > 0$. Consider

$$G = \left\{x \in X : \, x = \lim_{k\to\infty} \alpha(t_k)\right\}\,, \quad t_k \in [a,\,c)\,, \quad \lim_{k\to\infty} t_k = c\,.$$

Note that $c \neq b$. In fact, assume that $c = b$, then $|\beta| = f(|\alpha|)$ is compactum in $B(x_0,\varepsilon_0)$, and we obtain a contradiction.

Now, let $c \neq b$. Letting to subsequences, if it is need, we can restrict us by monotone sequences t_k. For $x \in G$, by continuity of f, $f\left(\alpha(t_k)\right) \to f(x)$ as $k \to \infty$, where $t_k \in [a,\,c)$, $t_k \to c$ as $k \to \infty$. However, $f\left(\alpha(t_k)\right) = \beta(t_k) \to \beta(c)$ as $k \to \infty$. Thus, f is a constant on G in $B(x_0,\varepsilon_0)$. From other hand, $\overline{\alpha}$ is a compact set, because $\overline{\alpha}$ is a closed subset of the compact space $\overline{B(x_0,\varepsilon_0)}$ (see [Ku, Theorem 2.II.4, § 41]). Now, by Cantor condition on the compact $\overline{\alpha}$, by monotonicity of $\alpha\left([t_k,\,c)\right)$,

$$G = \bigcap_{k=1}^{\infty} \overline{\alpha\left([t_k,\,c)\right)} \neq \varnothing\,,$$

see [Ku, 1.II.4, § 41]. Now, by [Ku, Theorem 5.II.5, § 47], $\overline{\alpha}$ is connected. By discreteness of f, G is a single-point set, and $\alpha\colon [a,\,c) \to B(x_0,\varepsilon_0)$ extends to a closed curve $\alpha\colon [a,\,c] \to K \subset B(x_0,\varepsilon_0)$, and $f\left(\alpha(c)\right) = \beta(c)$. By condition **A**, there exists a new maximal lifting α' of $\beta|_{[c,\,b)}$ starting in $\alpha(c)$. Uniting α and α', we obtain a new lifting α'' of β, which is defined in $[a, c')$, $c' \in (c,\,b)$, that contradicts to "maximality" of α. Thus, $d(\alpha(t), S(x_0,\varepsilon_0)) \to 0$ as $t \to c-0$.

Observe that $\Gamma(f(\overline{B(x_0,\varepsilon)}), \partial f(B(x_0,\varepsilon_0)), X') > f(\Gamma^*)$, and, consequently, by (2.3)

$$M_{\alpha'}\left(\Gamma(f(\overline{B(x_0,\varepsilon)}), \partial f(B(x_0,\varepsilon_0)), X')\right) \leqslant M_{\alpha'}\left(f(\Gamma^*)\right)\,. \qquad (3.3)$$

Consider

$$S_{\varepsilon} = S(x_0,\,\varepsilon)\,, \quad S_{\varepsilon_0} = S(x_0,\,\varepsilon_0)\,,$$

where ε_0 is from conditions of the lemma, and $\varepsilon \in (0,\,\varepsilon_0^{\prime})$. Since every curve $\alpha \in \Gamma^*$ satisfies the condition $d(\alpha(t), S(x_0,\varepsilon_0)) \to 0$ as $t \to c-0$, we obtain that $\Gamma\left(S_{\varepsilon}, S_{\varepsilon_0-\delta}, A(x_0,\varepsilon,\varepsilon_0-\delta)\right) < \Gamma^*$ at sufficiently small $\delta > 0$ and, consequently, $f(\Gamma\left(S_{\varepsilon}, S_{\varepsilon_0-\delta}, A(x_0,\varepsilon,\varepsilon_0-\delta)\right)) < f(\Gamma^*)$. Now

$$M_{\alpha'}\left(f(\Gamma^*)\right) \leqslant M_{\alpha'}\left(f\left(\Gamma\left(S_{\varepsilon}, S_{\varepsilon_0-\delta}, A(x_0,\varepsilon,\varepsilon_0-\delta)\right)\right)\right)\,. \qquad (3.4)$$

By (3.3) and (3.4),

$$M_{\alpha'}(\Gamma(f(\overline{B(x_0,\varepsilon)}), \partial f(B(x_0,\varepsilon_0)), X')) \leqslant M_{\alpha'}\left(f\left(\Gamma\left(S_{\varepsilon}, S_{\varepsilon_0-\delta}, A(x_0,\varepsilon,\varepsilon_0-\delta)\right)\right)\right)\,. \qquad (3.5)$$

Let $\eta(t)$ be an arbitrary nonnegative Lebesgue measurable function with the condition $\int\limits_{\varepsilon}^{\varepsilon_0} \eta(t)dt = 1$. Consider the family of function $\eta_{\delta}(t) = \frac{\eta(t)}{\int\limits_{\varepsilon}^{\varepsilon_0-\delta} \eta(t)dt}$. (Since $\int\limits_{\varepsilon}^{\varepsilon_0} \eta(t)dt = 1$, we can choose $\delta > 0$ such that $\int\limits_{\varepsilon}^{\varepsilon_0-\delta} \eta(t)dt > 0$). Since $\int\limits_{\varepsilon}^{\varepsilon_0-\delta} \eta_{\delta}(t)dt = 1$,

$$M_{\alpha'}\left(f\left(\Gamma\left(S_{\varepsilon}\,, S_{\varepsilon_0-\delta}, A(x_0,\varepsilon,\varepsilon_0-\delta)\right)\right)\right) \leqslant$$

$$\leqslant \frac{1}{\left(\int\limits_{\varepsilon}^{\varepsilon_0-\delta} \eta(t)dt\right)^{\alpha}} \int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\eta^{\alpha}(d(x,x_0))\ \ d\mu(x)\,. \tag{3.6}$$

Letting to the limit as $\delta\to 0$, by (3.5), we obtain that

$$M_{\alpha'}\left(f\left(\Gamma\left(S_{\varepsilon}\,, S_{\varepsilon_0}, A(x_0,\varepsilon,\varepsilon_0)\right)\right)\right)\leqslant \int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\eta^{\alpha}(d(x,x_0))\ \ d\mu(x)$$

for every nonnegative Lebesgue measurable function $\eta(t)$ with $\int\limits_{\varepsilon}^{\varepsilon_0}\eta(t)dt=1$. The desired conclusion follows now from the lemma 2.1. $\Box$

Denote $\mathfrak{L}_{x_0,Q,B_R,\delta,\mathbf{A}}(D)$ a family of all open discrete ring Q-mappings $f\colon D\to B_R\setminus K_f$ at $x_0\in D$ with $\mathbf{A}$-condition, where $B_R\subset X'$ is some fixed ball of a radius R, and K_f is some nondegenerate continuum in B_R with $\sup\limits_{x,y\in K_f} d'(x,y)\geqslant\delta>0$. A following statement is a main tool for a proof of equicontinuity result in a general situation.

Lemma 3.2. *Let D be a domain in a locally compact and locally connected metric space (X,d,μ) with a finite Hausdorff dimension $\alpha\geqslant 2$, and let (X',d',μ') be an Ahlfors α'-regular metric space which supports $(1;\alpha')$-Poincare inequality.*

Assume also that, (3.1) holds for some $\varepsilon_0'\in(0,\varepsilon_0)$ and some family of nonnegative Lebesgue measurable functions $\{\psi_{\varepsilon}(t)\}$, $\psi_{\varepsilon}\colon(\varepsilon,\varepsilon_0)\to(0,\infty)$, $\varepsilon\in(0,\varepsilon_0')$, where $F(\varepsilon,\varepsilon_0)$ satisfies the condition $F(\varepsilon,\varepsilon_0)=o(I^n(\varepsilon,\varepsilon_0))$ as $\varepsilon\to 0$, and $I(\varepsilon,\varepsilon_0)$ is defined by (2.9).

Now, $\mathfrak{L}_{x_0,Q,B_R,\delta,\mathbf{A}}(D)$ is equicontinuous at x_0.

Proof. Fix $f\in\mathfrak{L}_{x_0,Q,B_R,\delta,\mathbf{A}}(D)$. Set $A:=B(x_0,\varepsilon_0)\subset D$. Since X is locally connected and locally compact space, we can find a sequence $B(x_0,\varepsilon_k)$, $k=0,1,2,\ldots$, $\varepsilon_k\to 0$ as $k\to\infty$, such that $V_{k+1}\subset\overline{B(x_0,\varepsilon_k)}\subset V_k$, where V_k are continua in G. Observe that $f(V_k)$ are K_f continua in B_R, in fact, $f(V_k)$ is a continuum as continuous image of a continuum (see e.g. [Ku, Theorem 1.III.41 and Theorem 3.I.46]).

Note that $\Gamma(K_f,f(V_k),X')>\Gamma(f(V_k),\partial f(A),X')$ (see [Ku, Theorem 1.I.5, § 46]), so, by (2.3)

$$M_{\alpha'}(\Gamma(f(V_k),\partial f(A),X'))\geqslant M_{\alpha'}(\Gamma(K_f,f(V_k),X'))\,. \tag{3.7}$$

By Proposition 2.1

$$M_{\alpha'}(\Gamma(K_f,f(V_k),X')\geqslant\frac{1}{C_1}\cdot\frac{\min\{\operatorname{diam} f(V_k),\operatorname{diam} K_f\}}{R}\,. \tag{3.8}$$

By Lemma 3.1, $M_{\alpha'}(\Gamma(K_f,f(V_k),X')\to 0$ as $k\to\infty$ and, therefore, by (3.1) and (3.8) we obtain that

$$\min\{\operatorname{diam} f(V_k),\operatorname{diam} K_f\}=\operatorname{diam} f(V_k)$$

as $k\to\infty$. By (3.1) and (3.8) it follows that, for every $\sigma>0$ there exists $k_0=k_0(\sigma)$ such that

$$\operatorname{diam} f(C)\leqslant\sigma \tag{3.9}$$

for every $k \geqslant k_0(\sigma)$. Since $V_{k+1} \subset \overline{B(x_0, \varepsilon_k)} \subset V_k$, the inequality (3.9) holds in $\overline{B(x_0, \varepsilon_k)}$ as $k \geqslant k_0(\sigma)$. Set $\varepsilon(\sigma) := \varepsilon_{k_0}$. Finally, given $\sigma > 0$ there exists $\varepsilon(\sigma) > 0$ such that $d'(f(x), f(x_0)) < \sigma$ as $d(x, x_0) < \varepsilon(\sigma)$ for every $f \in \mathfrak{L}_{x_0,Q,B_R,\delta,\mathbf{A}}(D)$. So, $\mathfrak{L}_{x_0,Q,B_R,\delta,\mathbf{A}}(D)$ is equicontinuous at x_0. $\Box$

Denote $\mathfrak{L}_{x_0,Q,B_R,\delta,\mathbf{A}}(D)$ a family of all open discrete ring Q-mappings $f\colon D \to B_R \setminus K_f$ at $x_0 \in D$ with $\mathbf{A}$-condition, where $B_R \subset X'$ is some fixed ball of a radius R, and K_f is some nondegenerate continuum in B_R with $\sup\limits_{x,y \in K_f} d'(x, y) \geqslant \delta > 0$. Now, from Lemma 3.2 and Proposition 2.2, we obtain the following statement.

Theorem 3.1. *Let D be a domain in a locally compact and locally connected metric space (X, d, μ) with a finite Hausdorff dimension $\alpha \geqslant 2$, and let (X', d', μ') be an Ahlfors α'-regular metric space which supports $(1; \alpha')$-Poincare inequality.*

If $Q \in FMO(x_0)$, then $\mathfrak{L}_{x_0,Q,B_R,\delta,\mathbf{A}}(D)$ is equicontinuous at x_0.

Taking into account [RS, Corollary 4.1], by Lemma 3.2, we obtain the following.

Corollary 3.1. *A conclusion of Theorem* 3.1 *holds, if instead of condition $Q \in FMO(x_0)$ we require that*

$$\limsup_{\varepsilon \to 0} \frac{1}{\mu(B(x_0, \varepsilon))} \int\limits_{B(x_0,\varepsilon)} Q(x)\, d\mu(x) < \infty\,.$$

4 Removability of isolated singularities

A proof of the following lemma can be given by analogy with [RS, Lemma 8.1].

Lemma 4.1. *Let D be a domain in a metric space (X, d, μ) with a finite Hausdorff dimension $\alpha \geqslant 2$, and let (X', d', μ') be an Ahlfors α'-regular metric space which supports $(1; \alpha')$-Poincare inequality. Assume that, there exists $\varepsilon_0 > 0$ and a Lebesgue measurable function $\psi(t)\colon (0, \varepsilon_0) \to [0, \infty]$ with the following property: for every $\varepsilon_2 \in (0, \varepsilon_0]$ there exists $\varepsilon_1 \in (0, \varepsilon_2]$, such that*

$$0 < I(\varepsilon, \varepsilon_2) := \int\limits_{\varepsilon}^{\varepsilon_2} \psi(t)\, dt < \infty \qquad (4.1)$$

for every $\varepsilon \in (0, \varepsilon_1)$. Suppose also, that

$$\int\limits_{\varepsilon < d(x,x_0) < \varepsilon_0} Q(x) \cdot \psi^{\alpha}(d(x, x_0))\; dv(x) \;=\; o\left(I^{\alpha}(\varepsilon, \varepsilon_0)\right) \qquad (4.2)$$

holds as $\varepsilon \to 0$.

Let Γ be a family of all curves $\gamma(t)\colon (0, 1) \to D \setminus \{x_0\}$ obeying $\gamma(t_k) \to x_0$ for some $t_k \to 0$, $\gamma(t) \not\equiv x_0$. Then $M_{\alpha'}(f(\Gamma)) = 0$.

In particular, (4.1) holds provided that $\psi \in L^1_{loc}(0, \varepsilon_0)$ satisfies the condition $\psi(t) > 0$ for almost every $t \in (0, \varepsilon_0)$.

Proof. Note that
$$\Gamma > \bigcup_{i=1}^{\infty} \Gamma_i\,, \tag{4.3}$$
where Γ_i is a family of curves $\alpha_i(t)\colon (0,1)\to D\setminus\{x_0\}$ such that $\alpha_i(1)\in\{0<d(x,x_0)=r_i<\varepsilon_0\}$, and r_i is some sequence with $r_i\to 0$ as $i\to\infty$, and $\alpha_i(t_k)\to x_0$ as $k\to\infty$ for the same sequence $t_k\to 0$ as $k\to\infty$. Fix $i\geqslant 1$. By (4.1), $I(\varepsilon, r_i)>0$ for some $\varepsilon_1\in(0,r_i]$ and every $\varepsilon\in(0,\varepsilon_1)$. Now, observe that, for specified $\varepsilon>0$, the function
$$\eta(t)=\begin{cases}\psi(t)/I(\varepsilon,r_i), & t\in(\varepsilon,r_i),\\ 0, & t\in\mathbb{R}\setminus(\varepsilon,r_i)\end{cases}$$
satisfies (2.6) in $A(x_0,\varepsilon,r_i)=\{x\in X:\varepsilon<d(x,x_0)<r_i\}$. Since f is a ring Q-mapping at x_0, we obtain that
$$M_{\alpha'}\left(f\left(\Gamma\left(S(x_0,\varepsilon),\,S(x_0,r_i),\,A(x_0,\varepsilon,r_i)\right)\right)\right)\leqslant$$
$$\leqslant \int\limits_{A(x_0,\varepsilon,r_i)} Q(x)\cdot\eta^{\alpha}(d(x,x_0))\,d\mu(x)\leqslant \mathfrak{F}_i(\varepsilon), \tag{4.4}$$
where $\mathfrak{F}_i(\varepsilon)=\frac{1}{(I(\varepsilon,r_i))^{\alpha}}\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\,\psi^{\alpha}(d(x,x_0))\,d\mu(x)$. By (4.2), $\mathfrak{F}_i(\varepsilon)\to 0$ as $\varepsilon\to 0$. Observe that
$$\Gamma_i>\Gamma\left(S(x_0,\varepsilon),\,S(x_0,r_i),\,A(x_0,\varepsilon,r_i)\right) \tag{4.5}$$
for every $\varepsilon\in(0,\varepsilon_1)$. Thus, by (4.4) and (4.5), we obtain that
$$M_{\alpha'}(f(\Gamma_i))\leqslant\mathfrak{F}_i(\varepsilon)\to 0 \tag{4.6}$$
for every fixed $i=1,2,\ldots,$ as $\varepsilon\to 0$. However, the left-hand side of (4.6) does not depend on ε, that implies that $M_{\alpha'}(f(\Gamma_i))=0$. Finally, by (4.3) and subadditivity of modulus ([Fu, Theorem 1(b)]), we obtain that $M_{\alpha'}(f(\Gamma))=0$. $\Box$

A domain D is called *a locally linearly connected at* $x_0\in\partial D$, if for every neighborhood U of x_0 there exists a ball $B(x_0,r)$ centered at x_0 of some radius r in U such that $B(x_0,r)\cap D$ is linearly connected. The above definition slightly differs from the standard (see [Ku, I.6, § 49]). The following lemma provides the main tool for establishing equicontinuity in the most general situation.

Lemma 4.2. *Let $G:=D\setminus\{x_0\}$ be a domain in a locally compact metric space (X,d,μ) with a finite Hausdorff dimension $\alpha\geqslant 2$, where G is locally linearly connected at $x_0\in D$, and let (X',d',μ') be an Ahlfors α'-regular metric space which supports $(1;\alpha')$-Poincare inequality.*

Assume that, there exists $\varepsilon_0>0$ and a Lebesgue measurable function $\psi(t)\colon(0,\varepsilon_0)\to[0,\infty]$ with the following property: for every $\varepsilon_2\in(0,\varepsilon_0]$ there exists $\varepsilon_1\in(0,\varepsilon_2]$, such that (4.1) holds for every $\varepsilon\in(0,\varepsilon_1)$. Suppose also that, (4.2) holds as $\varepsilon\to 0$.

Let B_R be a fixed ball in X' such that $\overline{B_R}$ is compactum, and let K be a continuum in B_R. If an open discrete ring Q-mapping $f\colon D\setminus\{x_0\}\to B_R\setminus K$ at x_0 satisfies $\mathbf{A}$-condition, then f has a continuous extension to x_0.

Proof. Since $G=D\setminus\{x_0\}$ is locally linearly connected at $x_0\in D$, we can consider that $B(x_0,\varepsilon_0)\setminus\{x_0\}$ is connected. Assume the contrary, namely that the map has no limit at x_0. Since $\overline{B_R}$ is compactum, the limit set $C(f,x_0)$ is not empty. Thus, there exist two sequences x_j and и $x_j^{\prime}$ in $B(x_0,\varepsilon_0)\setminus\{x_0\}$, $x_j\to x_0,\quad x_j^{\prime}\to x_0$, such that $d^{\,\prime}\left(f(x_j),\,f(x_j^{\prime})\right)\geqslant a>0$ for all $j\in\mathbb{N}$. Set $r_j=\max\left\{d(x_j,x_0),\,d(x_j^{\prime},x_0)\right\}$. By locally linearly connectedness of G at x_0, we can consider that $\overline{B(x_0,r_j)}\setminus\{x_0\}$ is linearly connected. Now, x_j and $x_j^{\prime}$ can be joined by a closed curve C_j in $\overline{B(x_0,r_j)}\setminus\{x_0\}$.

Set $\Gamma_{f(E_j)}:=\Gamma(f(C_j),K,B_R)$. By Proposition 2.1, $\Gamma_{f(E_j)}\neq\varnothing$. Let Γ_j^* be the family of all maximal f-liftings of $\Gamma_{f(E_j)}$ starting at C_j, and lying in $B(x_0,\varepsilon_0)\setminus\{x_0\}$. Such the family is well-defined because $\mathbf{A}$ is satisfied.

Arguing as in the proof of Lemma 3.1, we can show that

$$\Gamma_j^*\ =\ \Gamma_{E_{j_1}}\cup\Gamma_{E_{j_2}}\,, \tag{4.7}$$

where $\Gamma_{E_{j_1}}$ is a family of all curves $\alpha(t)\colon[a,\,c)\to B(x_0,\varepsilon_0)\setminus\{x_0\}$ started at C_j for which $\alpha(t_k)\to x_0$ as $t_k\to c-0$ and some sequence $t_k\in[a,\,c)$, and $\Gamma_{E_{j_2}}$ is a family of all curves $\alpha(t)\colon[a,\,c)\to B(x_0,\varepsilon_0)\setminus\{x_0\}$ started at C_j for which $\operatorname{dist}(\alpha(t_k),\partial B(x_0,\varepsilon_0))\to 0$ as $t_k\to c-0$ and some sequence $t_k\in[a,\,c)$.

By (4.7),

$$M_{\alpha^{\prime}}\left(\Gamma_{f(E_j)}\right)\leqslant M_{\alpha^{\prime}}(f(\Gamma_{E_{j_1}}))\,+\,M_{\alpha^{\prime}}(f(\Gamma_{E_{j_2}}))\,. \tag{4.8}$$

By Lemma 4.1, $M_{\alpha^{\prime}}(f(\Gamma_{E_{j_1}}))=0$.

From other hand, we observe that $\Gamma_{E_{j_2}}>\Gamma(S(x_0,r_j),S(x_0,\varepsilon_0-\frac{1}{m}),A(x_0,r_j,\varepsilon_0-\frac{1}{m}))$ for sufficiently large $m\in\mathbb{N}$. Set $A_j=\{x\in X: r_j<d(x,x_0)<\varepsilon_0-\frac{1}{m}\}$ and

$$\eta_j(t)=\left\{\begin{array}{rr}\psi(t)/I(r_j,\varepsilon_0-\frac{1}{m}), & t\in(r_j,\,\varepsilon_0-\frac{1}{m}),\\ 0, & t\in\mathbb{R}\setminus(r_j,\,\varepsilon_0-\frac{1}{m}).\end{array}\right.$$

Now, we have that $\int\limits_{r_j}^{\varepsilon_0-\frac{1}{m}}\eta_j(t)dt=\frac{1}{I\left(r_j,\varepsilon_0-\frac{1}{m}\right)}\int\limits_{r_j}^{\varepsilon_0-\frac{1}{m}}\psi(t)dt=1$. Now, by definition of the ring Q-mapping at x_0 and by (4.8), we obtain that

$$M_{\alpha^{\prime}}(f(\Gamma_{E_j}))\leqslant\frac{1}{{I(r_j,\varepsilon_0-\frac{1}{m})}^{\alpha}}\int\limits_{r_j<d(x,x_0)<\varepsilon_0}Q(x)\,\psi^{\alpha}(d(x,x_0))\,d\mu(x)\,.$$

Letting to the limit at $m\to\infty$ here, we obtain that

$$M_{\alpha^{\prime}}(f(\Gamma_{E_j}))\leqslant\mathcal{S}(r_j):=\frac{1}{{I(r_j,\varepsilon_0)}^{\alpha}}\int\limits_{r_j<d(x,x_0)<\varepsilon_0}Q(x)\,\psi^{\alpha}(d(x,x_0))\,d\mu(x).$$

By (4.2), $\mathcal{S}(r_j) \to 0$ as $j \to \infty$, and by (4.8) we obtain that

$$M_{\alpha'}\left(\Gamma_{f(E_j)}\right) \to 0\,, \qquad j \to \infty\,. \tag{4.9}$$

From other hand, by Proposition 2.1, we obtain that

$$M_{\alpha'}(\Gamma_{f(E_j)}) \geqslant \frac{1}{C} \cdot \frac{\min\{\operatorname{diam} f(C_j), \operatorname{diam} K\}}{R} \geqslant \delta > 0 \tag{4.10}$$

because $d'\left(f(x_j), f(x_j')\right) \geqslant a > 0$ for all $j \in \mathbb{N}$ assumption made above. However, (4.10) contradicts with (4.9). The contradiction obtained above proves the theorem. $\Box$

The following statements can be obtained from Lemma 4.2 and Proposition 2.2.

Theorem 4.1. *Let $G := D\setminus\{x_0\}$ be a domain in a locally compact metric space (X, d, μ) with a finite Hausdorff dimension $\alpha \geqslant 2$, where G is locally linearly connected at $x_0 \in D$, and let (X', d', μ') be an Ahlfors α'-regular metric space which supports $(1;\alpha')$-Poincare inequality.*

Let B_R be a fixed ball in X' such that $\overline{B_R}$ is compactum, and let K be a continuum in B_R. If an open discrete ring Q-mapping $f\colon D \setminus \{x_0\} \to B_R \setminus K$ at x_0 satisfies **A***-condition, and $Q : D \to (0,\infty)$ has FMO at x_0, then f has a continuous extension to x_0.*

Taking into account [RS, Corollary 4.1], by Lemma 3.2, we obtain the following.

Corollary 4.1. *A conclusion of Theorem 4.1 holds, if instead of condition $Q \in FMO(x_0)$ we require that*

$$\limsup_{\varepsilon\to 0} \frac{1}{\mu(B(x_0,\varepsilon))} \int\limits_{B(x_0,\varepsilon)} Q(x)d\mu(x) < \infty\,.$$

The following results complement [RS, Theorem 10.2].

Theorem 4.2. *Let $G := D\setminus\{x_0\}$ be a domain in a locally compact metric space (X, d, μ) with a finite Hausdorff dimension $\alpha \geqslant 2$, where G is locally linearly connected at $x_0 \in D$, and let (X', d', μ') be an Ahlfors α'-regular metric space which supports $(1;\alpha')$-Poincare inequality.*

Let B_R be a fixed ball in X' such that $\overline{B_R}$ is compactum, and let K be a continuum in B_R. If $f\colon D \setminus \{x_0\} \to B_R \setminus K$ is a ring Q-homeomorphism at x_0, and $Q : D \to (0,\infty)$ has FMO at x_0, or $\limsup\limits_{\varepsilon\to 0} \frac{1}{\mu(B(x_0,\varepsilon))} \int\limits_{B(x_0,\varepsilon)} Q(x)d\mu(x) < \infty$, then f has a continuous extension to x_0.

5 Boundary behavior

Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 1$ in spaces (X, d, μ) and (X', d', μ'), and let $Q : G \to [0,\infty]$ be a measurable function. Following to [Sm], we

say that a mapping $f : G \to G'$ is a ring Q-mapping at a point $x_0 \in \partial G$ if the inequality

$$M_{\alpha'}(f(\Gamma(C_1, C_0, A)) \leqslant \int\limits_{A\cap G} Q(x)\eta^{\alpha}(d(x, x_0))d\mu(x)$$

holds for any ring

$$A = A(x_0, r_1, r_2) = \{x \in X : r_1 < d(x, x_0) < r_2\}, \quad 0 < r_1 < r_2 < \infty\,,$$

and any two continua $C_0 \subset \overline{B(x_0, r_1)}$, $C_1 \subset X \setminus B(x_0, r_2)$, and any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that (2.6) holds.

We say that the boundary of the domain G is *strongly accessible at a point* $x_0 \in \partial G$, if, for every neighborhood U of the point x_0, there is a compact set $E \subset G$, a neighborhood $V \subset U$ of the point x_0 and a number $\delta > 0$ such that

$$M_{\alpha}(\Gamma(E, F, G)) \geqslant \delta$$

for every continuum F in G intersecting ∂U and ∂V. We say that the boundary ∂G is *strongly accessible*, if the corresponding property holds at every point of the boundary. The following lemma holds.

Lemma 5.1. *Let D be a domain in a metric space (X, d, μ) with a finite Hausdorff dimension $\alpha \geqslant 2$, $\overline{D}$ is a compact, and let (X', d', μ') be a metric space with a finite Hausdorff dimension $\alpha' \geqslant 2$. Let $f : D \to X'$ be an open discrete ring Q-mapping at $b \in \partial D$, $f(D) = D'$, D is locally linearly connected at b, $C(f, \partial D) \subset \partial D'$, and D' is strongly accessible at least at one point $y \in C(f, b)$. Assume that*

$$0 < I(\varepsilon, \varepsilon_0) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)dt < \infty \tag{5.1}$$

for every $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 > 0$, and for some nonnegative Lebesgue measurable function $\psi(t)$, $\psi : (0, \varepsilon_0) \to (0, \infty)$. Assume that

$$\int\limits_{A(b,\varepsilon,\varepsilon_0)} Q(x) \cdot \psi^{\alpha}(d(x, b))\, d\mu(x) = o(I^{\alpha}(\varepsilon, \varepsilon_0))\,, \tag{5.2}$$

*where $A := A(b, \varepsilon, \varepsilon_0)$ is define in (2.5). If f satisfies **A**-condition, then $C(f, b) = \{y\}$.*

Proof. Assume the contrary. Now, there exist two sequences x_i, $x_i' \in D$, $i = 1, 2, \ldots$, obeying $x_i \to b$, $x_i' \to b$ as $i \to \infty$, $f(x_i) \to y$, $f(x_i') \to y'$ as $i \to \infty$ и $y' \neq y$. Observe that y and $y' \in \partial D'$, because $C(f, \partial D) \subset \partial D'$ by assumption of Lemma. By a definition of strong accessibility of a boundary at $y \in \partial D'$, for every neighborhood U of y, there exists a compact $C_0' \subset D'$, a neighborhood V of y, $V \subset U$, and $\delta > 0$ such that

$$M_{\alpha'}(\Gamma(C_0', F, D')) \geq \delta > 0 \tag{5.3}$$

for every compact F, intersecting ∂U and ∂V. By the assumption $C(f, \partial D)\subset \partial D^{\,\prime}$, $C_0\cap\partial D=\varnothing$ for $C_0:=f^{\,-1}(C_0^{\,\prime})$. Without loss of generalization, $C_0\cap \overline{B(b, \varepsilon_0)}=\varnothing$. Since D is locally linearly connected at b, we can join x_i and $x_i^{\,\prime}$ by a curve γ_i, which lies in $\overline{B(b, 2^{\,-i})}\cap D$. Since $f(x_i)\in V$ and $f(x_i^{\,\prime})\in D\setminus \overline{U}$ for sufficiently large $i\in {\Bbb N}$, by (5.3), there exists $i_0\in {\Bbb N}$ such that

$$M_{\alpha^{\,\prime}}(\Gamma(C_0^{\,\prime}, f(\gamma_i), D^{\,\prime}))\ge\delta>0 \tag{5.4}$$

for every $i\ge i_0\in {\Bbb N}$. Given $i\in {\Bbb N}$, $i\ge i_0$, consider a family $\Gamma_i^{\,\prime}$ of maximal f-liftings $\alpha_i(t):[a, c)\rightarrow D$ of $\Gamma(C_0^{\,\prime}, f(\gamma_i), D^{\,\prime})$ started at γ_i. (Such a family exists by condition $\mathbf{A}$). Since $C(f, \partial D)\subset \partial D^{\,\prime}$, we conclude that $\alpha_i(t)\in \Gamma_i^{\,\prime}$, $\gamma_i:[a, c)\rightarrow D$, does not tend to the boundary of D as $t\rightarrow c-0$. Now $C(\alpha_i(t), c)\subset D$. Since $\overline{D}$ is a compact, $C(\alpha_i(t), c)\ne\varnothing$.

Assume that $\alpha_i(t)$ has no limit at $t\rightarrow c-0$. We show that $C(\alpha_i(t), c)$ is a continuum in D. In fact, $C(\alpha_i(t), c)=\bigcap\limits_{k\,=\,1}^{\infty}\,\overline{\alpha\left([t_k,\, c)\right)}$, where t_k is increasing. By Cantor condition on the compact $\overline{\alpha}$, by monotonicity of $\alpha\left([t_k,\, c)\right)$,

$$G\,=\,\bigcap\limits_{k\,=\,1}^{\infty}\,\overline{\alpha\left([t_k,\, c)\right)}\ne\varnothing\,,$$

see [Ku, 1.II.4, § 41]. Now, G is connected as an intersection of countable collection of decreasing continua (see [Ku, Therorem 5, § 47(II)]).

So, $C(\alpha_i(t), c)$ is a continuum in D. By continuity of f, we obtain that $f\equiv const$ on $C(\alpha_i(t), c)$, which contradicts with discreteness of f.

Now, $\exists\lim\limits_{t\rightarrow c-0}\alpha_i(t)=A_i\in D$, and $c=b$. Now, we have that $\lim\limits_{t\rightarrow b-0}\alpha_i(t):=A_i$, and, simultaneously, by continuity of f in D,

$$f(A_i)=\lim\limits_{t\rightarrow b-0}f(\alpha_i(t))=\lim\limits_{t\rightarrow b-0}\beta_i(t)=B_i\in C_0^{\,\prime}\,.$$

It follows from the definition of C_0 that $A_i\in C_0$. We can immerse C_0 into some continuum $C_1\subset D$, see [Sm, Lemma 1]. We can consider that $C_1\cap\overline{B(b, \varepsilon_0)}=\varnothing$ by decreasing of $\varepsilon_0>0$. Putting $I(\varepsilon, \varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)dt$ we observe that the function

$$\eta(t)=\left\{\begin{array}{rr} \psi(t)/I(2^{-i}, \varepsilon_0), & t\in(2^{-i}, \varepsilon_0),\\ 0, & t\in {\Bbb R}\setminus (2^{-i}, \varepsilon_0)\,, \end{array} \right.$$

satisfies (2.6) at $r_1:=2^{-i}$, $r_2:=\varepsilon_0$. Now, by (5.1)–(5.2) and definition of the ring Q-mapping at the boundary point,

$$M_{\alpha^{\,\prime}}\left(f\left(\Gamma_i^{\,\prime}\right)\right)\le \Delta(i)\,, \tag{5.5}$$

where $\Delta(i)\rightarrow 0$ as $i\rightarrow\infty$. However, $\Gamma(C_0^{\,\prime}, F, D^{\,\prime})=f(\Gamma_i^{\,\prime})$, and by (5.5) we obtain that

$$M_{\alpha^{\,\prime}}(\Gamma(C_0^{\,\prime}, F, D^{\,\prime}))= M_{\alpha^{\,\prime}}\left(f(\Gamma_i^{\,\prime})\right)\le \Delta(i)\rightarrow 0 \tag{5.6}$$

as $i\rightarrow \infty$. However, (5.6) contradicts with (5.4). Lemma is proved. $\Box$

The following statements can be obtained from Lemma 5.1, Proposition 2.2 and [RS, Corollary 4.1].

Theorem 5.1. *Let D be a domain in a metric space (X,d,μ) with locally finite Borel measure μ and finite Hausdorff dimension $\alpha\geqslant 2$, $\overline{D}$ is a compact, and let (X',d',μ') be a metric space with locally finite Borel measure μ' and finite Hausdorff dimension $\alpha'\geqslant 2$. Let $f:D\to X'$ be an open discrete ring Q-mapping at $b\in\partial D$, $f(D)=D'$, D is locally linearly connected at b, $C(f,\partial D)\subset\partial D'$, and D' is strongly accessible at least at one point $y\in C(f,b)$. Assume that $Q\in FMO(b)$ and, simultaneously, Q obeying (2.16) at b. If f satisfies $\mathbf{A}$-condition, then $C(f,b)=\{y\}$.*

Theorem 5.2. *Let D be a domain in a metric space (X,d,μ) with locally finite Borel measure μ and finite Hausdorff dimension $\alpha\geqslant 2$, $\overline{D}$ is a compact, and let (X',d',μ') be a metric space with locally finite Borel measure μ' and finite Hausdorff dimension $\alpha'\geqslant 2$. Let $f:D\to X'$ be an open discrete ring Q-mapping at $b\in\partial D$, $f(D)=D'$, D is locally linearly connected at b, $C(f,\partial D)\subset\partial D'$, and D' is strongly accessible at least at one point $y\in C(f,b)$. Assume that $\limsup\limits_{\varepsilon\to 0}\frac{1}{\mu(B(x_0,\varepsilon))}\int\limits_{B(b,\varepsilon)}Q(x)\,d\mu(x)<\infty$ and, simultaneously, Q obeying (2.16) at b. If f satisfies $\mathbf{A}$-condition, then $C(f,b)=\{y\}$.*

6 Examples and open problems

Example 1. Now, let us to show that, the FMO condition can not be replaced by a weaker requirement $Q\in L^p$, $p\geqslant 1$, in Theorem 4.1 (see [MRSY, Proposition 6.3]). For simplicity, we consider a case $X=X'=\mathbb{R}^n$.

Theorem 6.1. *Given $p>1$, there exists $Q\in L^p(\mathbb{B}^n)$, $n\geq 2$, and bounded ring Q-homeomorphism $f:\mathbb{B}^n\setminus\{0\}\to\mathbb{R}^n$ at 0, for which $x_0=0$ is essential singularity.*

Proof. Set

$$f(x)=\frac{1+|x|^{\alpha}}{|x|}\cdot x\,,$$

where $\alpha\in(0,n/p(n-1))$. Without loss of generality, we can consider that $\alpha<1$. Observe that, f maps $\mathbb{B}^n\setminus\{0\}$ onto $\{1<|y|<2\}$ in $\mathbb{R}^n$, and $C(0,f)=\mathbb{S}^{n-1}$. Thus, $x_0=0$ is essential singularity.

Now, we show that f is a ring Q-homeomorphism at 0 and some $Q\in L^p(\mathbb{B}^n)$. Note that, f is a homeomorphism in $\mathbb{B}^n\setminus\{0\}$, and $f\in C^1\left(\mathbb{B}^n\setminus\{0\}\right)$. Now $f\in W_{loc}^{1,n}\left(\mathbb{B}^n\setminus\{0\}\right)$. Set

$$J(x,f)=\det f'(x),\quad l\left(f'(x)\right)\;=\;\min_{h\in\mathbb{R}^n\setminus\{0\}}\frac{|f'(x)h|}{|h|}\tag{6.1}$$

and

$$K_I(x,f)\quad=\quad\begin{cases}\frac{|J(x,f)|}{l(f'(x))^n}, & J(x,f)\neq 0,\\ 1, & f'(x)=0,\\ \infty, & \text{otherwise}\end{cases}\quad.\tag{6.2}$$

Then there exist systems of vectors $e_1,\ldots,e_n$ and $\widetilde{e_1},\ldots,\widetilde{e_n}$, and nonnegative numbers $\lambda_1(x_0),\ldots,\lambda_n(x_0)$, $\lambda_1(x_0)\leqslant\ldots\leqslant\lambda_n(x_0)$, such that $f'(x_0)e_i=\lambda_i(x_0)\widetilde{e_i}$ (see. [Re, 4.1.I]), and

$$|J(x_0,f)|=\lambda_1(x_0)\ldots\lambda_n(x_0),\quad l(f'(x_0))=\lambda_1(x_0)\,,$$
$$K_I(x_0,f)=\frac{\lambda_1(x_0)\ldots\lambda_n(x_0)}{\lambda_1^n(x_0)}\,.$$

Since f has a type $f(x)=\frac{x}{|x|}\rho(|x|)$, it is not difficult to show that, the "main vectors" $e_{i_1},\ldots,e_{i_n}$ and $\widetilde{e_{i_1}},\ldots,\widetilde{e_{i_n}}$ are $(n-1)$ linearly independent tangent vectors to $S(0,r)$ at x_0, where $|x_0|=r$, and one radial vector, which is orthogonal to them. We also can show that, in this case, the corresponding "stretchings", denoted as $\lambda_\tau(x_0)$ and λ_r, are $\lambda_\tau(x_0):=\lambda_{i_1}(x_0)=\ldots=\lambda_{i_{n-1}}(x_0)=\frac{\rho(r)}{r}$ и $\lambda_r(x_0):=\lambda_{i_n}=\rho'(r)$, correspondingly. From other hand, it is known that f is a ring Q-homeomorphism at $x_0=0$ under $Q=K_I(x,f)$ (see [MRSY, Theorem 8.6]).

Given $e\in\mathbb{S}^{n-1}$, observe that, $\frac{\partial f}{\partial e}(x_0)=\lim\limits_{t\to+0}\frac{f(x_0+te)-f(x_0)}{t}=\frac{\partial f}{\partial e}(x_0)=f'(x_0)e$ whenever x_0 is differentiability point of f. Let $\lambda_\tau(x_0)$ is a stretching, corresponding to a tangent direction at $x_0\in\mathbb{B}^n\setminus\{0\}$, and $\lambda_r(x_0)$ is a stretching, corresponding to a radial direction at x_0. Now

$$\lambda_\tau(x_0)=(1+|x_0|^\alpha)/|x_0|\,,\qquad\lambda_r(x_0)=\alpha|x_0|^{\alpha-1}\,.$$

Since $\lambda_\tau(x_0)\geq\lambda_r(x_0)$, we obtain that $l(f'(x_0))=\lambda_r(x_0)$. By (5.3), we have that

$$Q(x):=K_I(x_0,f)=\left(\frac{1}{\alpha}\right)^{n-1}\cdot\frac{(1+|x_0|^\alpha)^{n-1}}{|x_0|^{\alpha(n-1)}}\,. \tag{6.3}$$

For $r<1$,

$$Q(x)\leq\frac{C}{|x|^{\alpha(n-1)}}\,,\quad C:=\left(\frac{2}{\alpha}\right)^{n-1}\,.$$

Thus, we obtain that

$$\int\limits_{\mathbb{B}^n}(Q(x))^p\,dm(x)\leq C^p\int\limits_{\mathbb{B}^n}\frac{dm(x)}{|x|^{p\alpha(n-1)}}=$$

$$=C^p\int\limits_0^1\int\limits_{S(0,r)}\frac{d\mathcal{A}}{|x|^{p\alpha(n-1)}}dr=\omega_{n-1}C^p\int\limits_0^1\frac{dr}{r^{(n-1)(p\alpha-1)}}\,. \tag{6.4}$$

Since $I:=\int\limits_0^1\frac{dr}{r^\beta}$ is convergent at $\beta<1$, the integral in right-hand side of (6.4) is convergent, because $\beta:=(n-1)(p\alpha-1)$ satisfies $\beta<1$ at $\alpha\in(0,n/p(n-1))$.

Now, $Q(x)\in L^p(\mathbb{B}^n)$. $\Box$

Example 2. Now we show that the FMO condition can not be replaced by a weaker requirement $Q\in L^p$, $p\geqslant1$, in Theorems 2.1 and 3.1. We consider the case $X=X'=\mathbb{R}^n$, also.

Set $D:=\mathbb{B}^n\setminus\{0\}\subset\mathbb{R}^n$, $D':=B(0,2)\setminus\{0\}\subset\mathbb{R}^n$. Denote $\mathfrak{A}_Q$ a family of all ring Q-homeomorphisms $g:\mathbb{B}^n\setminus\{0\}\to\mathbb{R}^n$ at 0. The following statement holds.

Theorem 6.2. *Given $p \ge 1$, there exist $Q : \mathbb{B}^n \to [1, \infty]$, $Q(x) \in L^p(\mathbb{B}^n)$ and $g_m \in \mathfrak{A}_Q$ for which g_m has a continuous extension to $x_0 = 0$, however, $\{g_m(x)\}_{m=1}^{\infty}$ is not equicontinuous at $x_0 = 0$.*

Proof. Given $p \ge 1$ and $\alpha \in (0, n/p(n-1))$, $\alpha < 1$, we define $g_m : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ as

$$g_m(x) \;=\; \left\{ \begin{array}{rl} \frac{1+|x|^{\alpha}}{|x|} \cdot x\,, & 1/m \le |x| \le 1, \\ \frac{1+(1/m)^{\alpha}}{(1/m)} \cdot x\,, & 0 < |x| < 1/m\ . \end{array} \right.$$

Observe that, g_m maps $D = \mathbb{B}^n \setminus \{0\}$ onto $D' = B(0, 2) \setminus \{0\}$, and that $x_0 = 0$ is removable singularity for g_m, $m \in \mathbb{N}$. Moreover, $\lim\limits_{x \to 0} g_m(x) = 0$, and g_m is a constant as $|x| \ge 1/m$. In fact, $g_m(x) \equiv g(x)$ for $x : \frac{1}{m} < |x| < 1$, $m = 1, 2 \ldots$, where $g(x) = \frac{1+|x|^{\alpha}}{|x|} \cdot x$.

Observe $g_m \in ACL(\mathbb{B}^n)$. In fact, $g_m^{(1)}(x) = \frac{1+(1/m)^{\alpha}}{(1/m)} \cdot x$, $m = 1, 2, \ldots$, belongs to C^1 in $B(0, 1/m+\varepsilon)$ at sufficiently small $\varepsilon > 0$. From other hand, $g_m^{(2)}(x) = \frac{1+|x|^{\alpha}}{|x|} \cdot x$ are C^1-mappings in

$$A(1/m - \varepsilon, 1, 0) = \{x \in \mathbb{R}^n : 1/m - \varepsilon < |x| < 1\}$$

at small $\varepsilon > 0$. Thus g_m are lipschitzian in $\mathbb{B}^n$ and, consequently, $g_m \in ACL(\mathbb{B}^n)$ (see, e.g., [Va$_1$, sect. 5, p. 12]). As above, we obtain

$$K_I(x, g_m) = \left\{ \begin{array}{cl} \left(\frac{1+|x|^{\,\alpha}}{\alpha |x|^{\,\alpha}}\right)^{n-1}, & 1/m \le |x| \le 1, \\ 1\,, & 0 < |x| < 1/m\,. \end{array} \right.$$

Observe that $K_I(x, g_m) \le c_m$ for every $m \in \mathbb{N}$ and some constant. Now, $g_m \in W_{loc}^{1,n}(\mathbb{B}^n)$ and $g_m^{-1} \in W_{loc}^{1,n}(B(0, 2))$, because g_m and g_m^{-1} are quasiconformal (see, e.g., [Va$_1$, Corollary 13.3 and Theorem 34.6]). By [MRSY, Theorem 8.6], g_m are ring Q-homeomorphisms in $D = \mathbb{B}^n \setminus \{0\}$ at $Q = Q_m(x) := K_I(x, g_m)$. Moreover, g_m are Q-homeomorphisms with $Q = \left(\frac{1+|x|^{\,\alpha}}{\alpha |x|^{\,\alpha}}\right)^{n-1}$. Since $\alpha p(n-1) < n$, we have $Q \in L^p(\mathbb{B}^n)$, see proof of the theorem 6.1. From another hand, we have that

$$\lim_{x \to 0} |g(x)| = 1\,, \tag{6.5}$$

and g maps $\mathbb{B}^n \setminus \{0\}$ onto $1 < |y| < 2$. By (6.5), we obtain that

$$|g_m(x)| = |g(x)| \ge 1 \qquad\qquad \forall \quad x : |x| \ge 1/m, \quad m = 1, 2, \ldots\,,$$

i.e. $\{g_m\}_{m=1}^{\infty}$ is not equicontinuous a the origin. □

Open problem 1. If $X = X' = \mathbb{R}^n$, for all $\beta : [a,\, b) \to X'$ and $x \in f^{-1}(\beta(a))$, an open discrete mapping f has a maximal f-lifting starting at x. *To describe properties of the metric spaces (X, d, μ) and (X', d', μ'), for which, for every curve $\beta : [a,\, b) \to X'$ and $x \in f^{-1}(\beta(a))$, there exists a maximal f-lifting starting at x under every open discrete mapping $f : X \to X'$.*

Open problem 2. We say that the path connected space (X, d, μ) is weakly flat at a point $x_0 \in X$ if, for every neighborhood U of the point x_0 and every number $P > 0$, there

is a neighborhood $V\subseteq U$ of x_0 such that $M_\alpha(\Gamma(E,F,X))\geqslant P$ for any continua E and F in X intersecting ∂V and ∂U. We say that a space (X,d,μ) is weakly flat, if it is weakly flat at every point. *To find relationship between weakly flat spaces and spaces, which are Ahlfors α-regular and support $(1;\alpha)$-Poincare inequality.*

References

[AS] T. Adamowicz and N. Shanmugalingam, "Non-conformal Loewner type estimates for modulus of curve families", *Ann. Acad. Sci. Fenn. Math.*, **35** (2010), 609—626.

[AC] C. Andreian Cazacu, "On the length-area dilatation", *Complex Var. Theory Appl.*, 50(2005), 765–776.

[Cr_1] M. Cristea, "Local homeomorphisms having local ACL^n inverses", *Complex Variables and Elliptic Equations*, **53**:1 (2008), 77–99.

[Cr_2] M. Cristea, "Open discrete mappings having local ACL^n inverses", *Complex Variables and Elliptic Equations*, **55**:1–3 (2010), 61–90.

[Ge] F.W. Gehring, "The definitions and exceptional sets for quasiconformal mappings", *Ann. Acad. Sci. Fenn. Ser. A.*, 281 (1960), 1–28.

[Gol_1] A. Golberg, "Homeomorphisms with finite mean dilatations", *Contemporary Math.*, **382** (2005), 177–186.

[Gol_2] A. Golberg, "Differential properties of (α,Q)–homeomorphisms", *Further Progress in Analysis, World Scientific Publ.*, (2009), 218–228.

[Fu] B. Fuglede, "Extremal length and functional completion", *Acta Math.*, 98 (1957), 171–219.

[GRSY] V. Ya. Gutlyanskii, V. I. Ryazanov, U. Srebro, E. Yakubov, *The Beltrami Equation: A Geometric Approach.* Developments in Mathematics, vol. 26, New York etc.: Springer, 2012.

[He] J. Heinonen, *Lectures on Analysis on metric spaces*, New York: Springer Science+Business Media, 2001.

[HK] S. Hencl, P. Koskela, *Lectures on mappings of finite distortion*, Lecture Notes in Mathematics, 2096, Cham: Springer, 2014.

[HW] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton: Princeton Univ. Press, 1948.

[IM] T. Iwaniec and G. Martin, *Geometric Function Theory and Non–Linear Analysis*, Oxford: Clarendon Press, 2001.

[IR_1] A. Ignat'ev and V. Ryazanov, "Finite mean oscillation in mapping theory" (Russian), *Ukr. Mat. Visn.*, 2 (2005), no. 3, 395–417, translation in *Ukr. Math. Bull.*, 2 (2005), no. 3, 403—424.

[IR$_2$] A. Ignat'ev and V. Ryazanov, "On the theory of the boundary behavior of spatial mappings" (Russian), *Ukr. Mat. Visn.*, 3 (2006), no. 2, 199–211, translation in *Ukr. Math. Bull.*, 3 (2006), no. 2, 189—201.

[Ku] K. Kuratowski, *Topology,* v. 2, New York–London: Academic Press, 1968.

[MRSY] O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, *Moduli in Modern Mapping Theory*, New York: Springer Science + Business Media, LLC, 2009.

[MRV$_1$] O. Martio, S. Rickman, J. Väisälä, "Definitions for quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A I. Math.*, **448** (1969), 1–40.

[MRV$_2$] O. Martio, S. Rickman, J. Väisälä, "Distortion and singularities of quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A1*, 465 (1970), 1–13.

[MRV$_3$] O. Martio, S. Rickman, J. Väisälä, "Topological and metric properties of quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A1.*, 488 (1971), 1–31.

[Na] R. Näkki, "Boundary behavior pf quasiconformal mappings in n-space", *Ann. Acad. Sci. Fenn. Ser. A1. Math.*, 484 (1970), 1–50.

[NP] R. Näkki and B. Palka, "Uniform equicontinuity of quasiconformal mappings", *Proc. Amer. Math. Soc.*, 37 (1973), no. 2, 427–433.

[Ra] K. Rajala, "Mappings of finite distortion: removable singularities for locally homeomorphic mappings", *Proc. Amer. Math. Soc.*, 132 (2004), no. 11, 3251–3258.

[Re] Yu.G. Reshetnyak, *Space mappings with bounded distortion*, English transl.: Amer. Math. Soc., Providence, RI, 1989.

[Ri] S. Rickman, *Quasiregular mappings*, Berlin etc.: Springer-Verlag, 1993.

[RS] V. Ryazanov, R. Salimov, "Weakly planar spaces and boundaries in the theory of mappings" (Russian), *Ukr. Mat. Visn.*, 4 (2007), no. 2, 199–234, translation in *Ukr. Math. Bull.*, 4 (2007), no. 2, 199–234

[Sev$_1$] E.A. Sevost'yanov, "Theory of moduli, capacities and normal families of mappings admitting a branching" (in Russian), *Ukr. Mat. Visnyk*, 4 (2007), no. 4, 582–604; translation in *Ukr. Math. Bull.*, 4 (2007), no. 4, 573–593.

[Sev$_2$] E.A. Sevost'yanov, "On the theory of the removal of singularities for mappings with an unbounded characteristic of quasiconformality" (Russian), *Izv. AN Rossii, Ser. Mat.*, 74 (2010), no. 1, 159–174, translation in *Izv. Math.*, 74 (2010), no. 1, 151–165.

[Sev$_3$] E.A. Sevost'yanov, "On boundary behavior of the open discrete mappings with unbounded characteristics" (in Russian), *Ukrainian Math. Zh.*, 64 (2012), no. 6, 855–859, translation in *Ukrainian Math. J.*, 64 (2012), no. 6, 979-984.

[Sm] E.S. Smolovaya, "Boundary behavior of ring Q-homeomorphisms in metric spaces", *Ukrainian Math. J.*, 62 (2010), no. 5, 785—793.

[Va$_1$] J. Väisälä, *Lectures on n – Dimensional Quasiconformal Mappings*, Lecture Notes in Math., 229, Berlin etc.: Springer–Verlag, 1971.

[Va$_2$] J. Väisälä, "Modulus and capacity inequalities for quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A 1 Math.*, 509 (1972), p. 1–14.

[Vo$_1$] S.K. Vodop'yanov, "Closedness of classes of mappings with bounded distortion on Carnot groups" (Russian), *Mat. Tr.*, 5 (2002), no. 2, p. 92–137.

[Vo$_2$] S.K. Vodopyanov, "Foundations of the Theory of Mappings with Bounded Distortion on Carnot Groups, in book "The Interaction of Analysis and Geometry", Contemporary Mathematics, AMS, 424 (2007), p. 303—344.

[VG] S.K. Vodop'yanov, V.M. Gol'dtshein, *Prostranstva Soboleva i spetsial'nye klassy otobrazhenii* (Russian) [Sobolev spaces and special classes of mappings], Novosibirsk: Novosibirsk. Gos. Univ., 1981.

[VGR$_1$] S.K. Vodop'yanov, V.M. Gol'dshtein, Yu.G. Reshetnyak, "The geometric properties of functions with generalized first derivatives" (Russian), *Uspekhi Mat. Nauk*, 34 (1979), no. 1(205), p. 17–65.

[VGR$_2$] S.K. Vodop'yanov, A.V. Greshnov, "Analytic properties of quasiconformal mappings on Carnot groups", *Siberian Mathematical Journal*, 36 (1995), no. 6, p. 1142–1151.

[UV] S.K. Vodop'yanov, A.D. Ukhlov, "Sobolev spaces and (P,Q)-quasiconformal mappings of Carnot groups", *Sibirsk. Mat. Zh.*, 39 (1998), no. 4, p. 776-–795.

[Vu] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Math., 1319, Berlin etc.: Springer–Verlag, 1988.

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